

A SPACETIME ESTIMATE FOR HARD SPHERES

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ABSTRACT. We review a spacetime estimate which bounds the strength of interaction for a gas of N hard spheres (billiard balls) dispersing into vacuum. The estimate controls a trace norm for a problem wherein traces have no meaning *a priori*, which makes the estimate particularly difficult to handle. We refer to other work for a rigorous proof; the present paper provides an alternative semi-heuristic derivation.

1. INTRODUCTION

The problem of interest to us is that of deriving various nonlinear partial differential equations (PDEs) starting from the Newtonian gas of N hard spheres. Depending on the chosen scalings, the relevant PDE could be the Navier-Stokes equations, Boltzmann's equation, etc. The existence and uniqueness of solutions to nonlinear PDEs is generally an open problem, except in the presence of very special conservation principles or perturbative assumptions. Even when solutions are known, the analysis tends to be quite complicated, depending on the strength of available *a priori* estimates for a hypothetical solution. For this reason, we are naturally led to the problem of deriving analogous *a priori* bounds on the particle model.

The main focus of this work is Proposition 4.1, which is a virial-type spacetime estimate for hard spheres. Virial-type estimates are essentially classical in the billiards literature, and they play a prominent role in the derivation of Boltzmann's equation. [6] The main difference with Proposition 4.1 is that the bound controls a quantity closely associated with the hard sphere BBGKY hierarchy, with no "smallness" conditions imposed on the initial data.

Section 2 introduces the basic notation of this work, which mostly follows the presentation of [4]. Section 3 gives an elementary derivation of an identity due to Illner [5]. In Section 4, we apply Illner's identity in a heuristic manner to derive the virial-type spacetime estimate; a rigorous proof may be found in [3].

2. NOTATION

Consider N hard spheres centered at positions $x_i \in \mathbb{R}^d$ with velocities $v_i \in \mathbb{R}^d$ for $i = 1, 2, \dots, N$. The spheres are considered to have identical mass and radius, and are in all other ways physically indistinguishable. For convenience, we will assume without loss that all particles have unit

diameter. The collection of all positions is a tuple X_N ,

$$X_N = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{dN}$$

and the velocities are similarly denoted

$$V_N = (v_1, v_2, \dots, v_N) \in \mathbb{R}^{dN}$$

The classical phase-space coordinates of i th particle are given by $z_i = (x_i, v_i)$, and the phase-space coordinates of the whole gas are denoted

$$Z_N = (z_1, z_2, \dots, z_N) \in \mathbb{R}^{2dN}$$

We may also write $Z_N = (X_N, V_N)$. The following function will play a central role in our analysis: for $t \in \mathbb{R}$ and $Z_N \in \mathbb{R}^{2dN}$, we define

$$r_N(t, Z_N) = \sum_{i=1}^N (x_i \cdot v_i - |v_i|^2 t) \quad (1)$$

Finally we may introduce the N -particle phase space \mathcal{D}_N , which is defined by

$$\mathcal{D}_N = \left\{ Z_N \in \mathbb{R}^{2dN} \mid \forall 1 \leq i < j \leq N, |x_i - x_j| > 1 \right\} \quad (2)$$

The choice of \mathcal{D}_N is motivated by requirement that the spheres be mutually disjoint at all times. The closure of \mathcal{D}_N in \mathbb{R}^{2dN} in the standard topology is denoted $\overline{\mathcal{D}}_N$, and we will also write $\partial\mathcal{D}_N = \overline{\mathcal{D}}_N \setminus \mathcal{D}_N$. We will use the notation a.e. $Z_N \in \mathcal{D}_N$ to refer to a typical point for the Lebesgue measure on \mathcal{D}_N . The notation a.e. $Z_N \in \partial\mathcal{D}_N$ will refer to a typical point for the induced surface measure arising from the natural embedding $\partial\mathcal{D}_N \subset \mathbb{R}^{2dN}$.

Formally speaking, we wish to solve Newton's laws with a hard core interaction. This means if $Z_N(t_0) = (X_N(t_0), V_N(t_0)) \in \mathcal{D}_N$ then

$$\begin{aligned} \left. \frac{d}{dt} X_N(t) \right|_{t=t_0} &= V_N(t_0) \\ \left. \frac{d}{dt} V_N(t) \right|_{t=t_0} &= 0 \end{aligned}$$

Hence the particles move freely between collisions. At each collision (that is, $Z_N(t_0) \in \partial\mathcal{D}_N$), the particles are required to interact elastically, thereby conserving momentum, energy, and angular momentum. The set of possible interactions for two-body elastic collisions is easy to parametrize explicitly. Suppose that there exists $i < j$ such that $x_j(t_0) = x_i(t_0) + \omega$ for some $\omega \in \mathbb{S}^{d-1}$; and, further suppose that $|x_{j'}(t_0) - x_{i'}(t_0)| > 1$ for any $i' < j'$ such that $(i', j') \neq (i, j)$. Let us denote

$$\begin{aligned} \lim_{t \rightarrow t_0^-} V_N(t) &= (v_1, \dots, v_i, \dots, v_j, \dots, v_N) \\ \lim_{t \rightarrow t_0^+} V_N(t) &= (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N) \end{aligned}$$

Then we have

$$\begin{aligned} v_i^* &= v_i + \omega \omega \cdot (v_j - v_i) \\ v_j^* &= v_j - \omega \omega \cdot (v_j - v_i) \end{aligned}$$

Remark. Similarly for a.e. $Z_N \in \partial\mathcal{D}_N$ we will use the notation Z_N^* to refer to the image of the point Z_N through the collision transformation. The map $Z_N \mapsto Z_N^*$ is a measurable involution.

In the above “definition,” we have neglected to specify uniquely what happens when more than two particles collide at the same time. Multiple particle interactions occur with zero probability, though this statement requires justification which we will not discuss. The hard sphere flow at time t defines a measurable map

$$\psi_N^t : \mathcal{D}_N \rightarrow \mathcal{D}_N$$

For each $t \in \mathbb{R}$, the map ψ_N^t preserves the Lebesgue measure on $\mathcal{D}_N \subset \mathbb{R}^{2dN}$. Complete proofs of the existence of the hard sphere flow ψ_N^t may be found in the literature. [1, 4]

Following Boltzmann’s great insight, we realize that it is not very interesting to discuss any *particular* trajectory $\{\psi_N^t Z_N\}_{t \geq 0}$, because it is physically infeasible (or impossible) to measure the positions and velocities of all the particles at a given instant. Therefore, the initial value problem for Newton’s laws is *not* the correct problem for us to solve. The correct approach is to place a probability density $f_N(0, Z_N)$ on the set of possible initial states $Z_N \in \mathcal{D}_N$. The function $f_N(0, Z_N)$ represents our uncertainty about the actual configuration of the particles. Since we have no physical means to distinguish between two particles in our model, the function $f_N(0, Z_N)$ must be *symmetric* with respect to interchange of particle indices.

We will denote by \mathcal{S}_N the symmetric group on N letters. Any permutation $\sigma \in \mathcal{S}_N$ acts on the phase-space coordinates $Z_N = (z_1, z_2, \dots, z_N) \in \mathcal{D}_N$ as follows:

$$\sigma Z_N = (z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(N)}) \in \mathcal{D}_N$$

Similarly, if $f_N(Z_N)$ is any function on \mathcal{D}_N , then σ acts on f_N by composition: $\sigma f_N = f_N \circ \sigma$. Let $\mathcal{P}(\mathcal{D}_N)$ denote the set of probability measures on \mathcal{D}_N , and furthermore let $\mathcal{P}_{\text{a.c.}}(\mathcal{D}_N)$ denote the set of probability measures which are absolutely continuous with respect to the Lebesgue measure on \mathcal{D}_N . Any element of $\mathcal{P}_{\text{a.c.}}(\mathcal{D}_N)$ may be represented uniquely (a.e. $Z_N \in \mathcal{D}_N$) by a non-negative function $f_N(Z_N)$ such that $\int_{\mathcal{D}_N} f_N(Z_N) dZ_N = 1$. Finally let $\mathcal{P}_{\text{a.c.}}^{\text{sym}}(\mathcal{D}_N)$ be the set of absolutely continuous measures on \mathcal{D}_N such that the associated function f_N is invariant under the action of \mathcal{S}_N . Henceforth, when we write f_N , we will always mean an element of $\mathcal{P}_{\text{a.c.}}^{\text{sym}}(\mathcal{D}_N)$.

Let $f_N(0)$ be any element of $\mathcal{P}_{\text{a.c.}}^{\text{sym}}(\mathcal{D}_N)$, which we regard as the initial state of the N particle gas. For any $t \in \mathbb{R}$ we will let $f_N(t)$ be the push-forward of $f_N(0)$ under the hard sphere flow ψ_N^t ; then, $f_N(t)$ is likewise an element of $\mathcal{P}_{\text{a.c.}}^{\text{sym}}(\mathcal{D}_N)$. Since ψ_N^t preserves the Lebesgue measure on \mathcal{D}_N ,

we may write the following expression for $f_N(t)$:

$$f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N) \quad (3)$$

The functions $f_N(0)$ and $f_N(t)$ may be extended by zero so as to be defined on \mathbb{R}^{2dN} .

It turns out that $f_N(t)$ still contains too much information to be a useful description of the state of the gas. Therefore, for any $1 \leq s \leq N$, we define the marginal $f_N^{(s)}(t)$ by partial integration:

$$f_N^{(s)}(t, Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_N) dz_{s+1} \dots dz_N \quad (4)$$

The evolution of the marginals $f_N^{(s)}(t)$ may be described explicitly via the so-called BBGKY hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon) [4], though we will *not* be making any use of the BBGKY hierarchy in this work. The marginals $f_N^{(s)}(t)$ are non-negative symmetric functions on \mathbb{R}^{2ds} with unit mass.

The spacetime estimate will control the trace of the marginals $f_N^{(s)}(t)$ along a certain hypersurface, with at most polynomial growth in N for large values of N . This is slightly problematic because the trace of an L^1 function is simply not defined; moreover, even if the data $f_N(0)$ is *smooth*, the function $f_N(t)$ typically develops singularities. Nevertheless, due to technical arguments which we will not discuss, it is possible to show that if $f_N(0)$ is smooth then the required traces of $f_N^{(s)}(t)$ do, in fact, exist. See [2, 4, 7, 8] for more information on regularity issues for hard spheres. Our estimates do not depend on the choice of regularization, except insofar as the regularized marginals must be a sequence of symmetric non-negative functions which are indeed marginals in the sense of (4).

3. ILLNER'S IDENTITY

Let us fix an initial point $Z_N \in \mathcal{D}_N$ in the microscopic phase-space, and consider the trajectory $\{\psi_N^t Z_N\}_{t \in \mathbb{R}}$. Our analysis begins with a simple observation: with $r_N(t, Z_N)$ as in (1), if we define

$$r_{Z_N}(t) = r_N(t, \psi_N^t Z_N) \quad (5)$$

then for any t_0 such that $\psi_N^{t_0} Z_N \in \mathcal{D}_N$ we have

$$\left. \frac{d}{dt} r_{Z_N}(t) \right|_{t=t_0} = 0 \quad (6)$$

Indeed, we see that if $\dot{x}_i = v_i$ and $\dot{v}_i = 0$ then

$$\frac{d}{dt} (x_i \cdot v_i - |v_i|^2 t) = 0$$

Therefore, the difference $r_{Z_N}(t) - r_{Z_N}(0)$ is simply equal to a sum along *collisions* of incremental jumps in $r_{Z_N}(t)$. It will turn out that all of these

jumps have the *same sign*, and we can write the jumps explicitly to yield Illner's identity.

Let us compute the jump in $r_{Z_N}(t)$ across a collision taking place at time $t_0 \in \mathbb{R}$. We may assume that the interacting particles are simply those labelled $i = 1, 2$, since collisions are binary and particles are indistinguishable. The position coordinates are continuous in time, so we write them x_1, x_2 , with $x_2 = x_1 + \omega$ for some $\omega \in \mathbb{S}^{d-1}$. The pre-collisional velocities will be denoted $v_1 \equiv v_1(t_0^-), v_2 \equiv v_2(t_0^-)$ and the post-collisional velocities will be denoted $v_1^* \equiv v_1(t_0^+), v_2^* \equiv v_2(t_0^+)$. We have

$$\begin{aligned} r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) &= \{ (x_1 \cdot v_1^* - |v_1^*|^2 t_0) + (x_2 \cdot v_2^* - |v_2^*|^2 t_0) \} + \\ &\quad + \{ - (x_1 \cdot v_1 - |v_1|^2 t_0) - (x_2 \cdot v_2 - |v_2|^2 t_0) \} \end{aligned}$$

Due to energy conservation,

$$|v_1^*|^2 + |v_2^*|^2 = |v_1|^2 + |v_2|^2$$

so we may eliminate the explicit dependence on t_0 .

$$r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) = x_1 \cdot v_1^* + x_2 \cdot v_2^* - x_1 \cdot v_1 - x_2 \cdot v_2$$

Since $x_2 = x_1 + \omega$, this gives us

$$r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) = x_1 \cdot (v_1^* + v_2^* - v_1 - v_2) + \omega \cdot (v_2^* - v_2)$$

Due to momentum conservation,

$$v_1^* + v_2^* = v_1 + v_2$$

so we may eliminate the explicit dependence on the position coordinates. Hence

$$r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) = \omega \cdot (v_2^* - v_2)$$

This is the same as

$$r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) = -\omega \cdot (v_2 - v_1)$$

by the collisional change of variables from Section (2). But v_1, v_2 are the velocities of the two particles *coming into* a collision, so we must have

$$(x_2 - x_1) \cdot (v_2 - v_1) \leq 0$$

and therefore $\omega \cdot (v_2 - v_1) \leq 0$. Hence,

$$r_{Z_N}(t_0^+) - r_{Z_N}(t_0^-) = |\omega \cdot (v_2 - v_1)| \geq 0$$

Adding up all collisions along the trajectory we obtain Illner's identity: [5]

Proposition 3.1. (*Illner's identity*) For a.e. $Z_N \in \mathcal{D}_N$ and a.e. $t \geq 0$ there holds

$$r_{Z_N}(t) - r_{Z_N}(0) = \sum_k |\omega_k \cdot (v_{j_k}(t_k^-) - v_{i_k}(t_k^-))| \quad (7)$$

where the sum \sum_k is over all collisions along the trajectory $\{\psi_N^\tau Z_N\}_\tau$ for $0 \leq \tau \leq t$.

We will require an auxiliary lemma of Illner & Pulvirenti which follows easily from Illner's identity. [6] In order to state the lemma, we introduce a new function on \mathcal{D}_N ,

$$I_N(Z_N) = \sum_{i=1}^N |x_i|^2 \quad (8)$$

The proof is a computation, which we include for completeness.

Lemma 3.2. *For a.e. $Z_N = (X_N, V_N) \in \mathcal{D}_N$ and all $t \in \mathbb{R}$, we have*

$$I_N(\psi_N^t Z_N) \geq I_N((X_N + V_N t, V_N)) \quad (9)$$

Proof. By time-reversibility we may assume $t \geq 0$. The function $I_N(\psi_N^t Z_N)$ is globally continuous in t for a.e. $Z_N \in \mathcal{D}_N$. With this in mind, it suffices to point out that the desired inequality is true for $t = 0$, and between collisions we have

$$\frac{d}{dt} \{I_N(\psi_N^t Z_N) - I_N((X_N + V_N t, V_N))\} = 2 \{r_{Z_N}(t) - r_{Z_N}(0)\}$$

We conclude by Proposition 3.1. \square

The next lemma is technical, and again follows from Illner's identity.

Lemma 3.3. *For a.e. $Z_N = (X_N, V_N) \in \mathcal{D}_N$, all $t \geq 0$, and all $\lambda > 0$, there holds*

$$|r_{Z_N}(t)| \leq \frac{1}{2} \lambda^{-1} \sum_{i=1}^N (\lambda^2 |x_i|^2 + |v_i|^2) \quad (10)$$

Proof. Recall that $r_{Z_N}(t) \equiv r_N(t, \psi_N^t Z_N)$. On the other hand, by (1),

$$\begin{aligned} |r_N(t, Z_N)| &\leq \sum_{i=1}^N |x_i \cdot v_i - |v_i|^2 t| \\ &= \sum_{i=1}^N |(x_i - v_i t) \cdot v_i| \\ &\leq \frac{1}{2} \sum_{i=1}^N (\lambda |x_i - v_i t|^2 + \lambda^{-1} |v_i|^2) \\ &= \frac{1}{2} \lambda I_N((X_N - V_N t, V_N)) + \frac{1}{2} \lambda^{-1} \sum_{i=1}^N |v_i|^2 \end{aligned}$$

We can bound the first term on the last line using Lemma 3.2. Hence,

$$|r_N(t, Z_N)| \leq \frac{1}{2} \lambda I_N(\psi_N^{-t} Z_N) + \frac{1}{2} \lambda^{-1} \sum_{i=1}^N |v_i|^2$$

Replace Z_N by $\psi_N^t Z_N$ on both sides and use the conservation of energy to conclude. \square

Combining Proposition 3.1 and Lemma 3.3, we obtain:

Corollary 3.4. *For a.e. $Z_N = (X_N, V_N) \in \mathcal{D}_N$ and all $\lambda > 0$, we have*

$$\sum_k |\omega_k \cdot (v_{j_k}(t_k^-) - v_{i_k}(t_k^-))| \leq 2\lambda^{-1} \sum_{i=1}^N (\lambda^2 |x_i|^2 + |v_i|^2) \quad (11)$$

where the sum \sum_k is over all collisions along the trajectory $\{\psi_N^t Z_N\}_{t \in \mathbb{R}}$.

4. AN AVERAGING TRICK

The previous section was primarily concerned with the number of collisions which occur along a single trajectory $\{\psi_N^t Z_N\}_t$. However, as has been explained in Section 2, we are really interested in ensemble averages over many trajectories. This is due to the physical fact that we cannot say with any precision what the initial state Z_N “really” is. We will prove the spacetime estimate by averaging both sides of (11) with respect to the *same* measure $f_N(0, Z_N) dZ_N$ and applying a change of variables on the left-hand side. The change of variables as presented here is not entirely rigorous, though we are confident that this approach can be converted into a rigorous proof. An alternative, completely rigorous, proof of the virial-type estimate has already been given. [3]

We will find it helpful to define an auxiliary function,

$$W_N^{(i,j)}(Z_N) = |(x_j - x_i) \cdot (v_j - v_i)| \quad (12)$$

Observe that $W_N^{(i,j)}(Z_N) = |\omega \cdot (v_j - v_i)|$ if $Z_N \in \partial\mathcal{D}_N$ represents a collision between particles i and j with $x_j = x_i + \omega$. For any $Z_N \in \mathcal{D}_N$ let $\tilde{i}_N(Z_N)$, $\tilde{j}_N(Z_N)$ be chosen such that $|x_{\tilde{i}_N} - x_{\tilde{j}_N}| \leq |x_i - x_j|$ for all $i \neq j$. This uniquely defines \tilde{i}_N, \tilde{j}_N for a.e. $Z_N \in \mathcal{D}_N$ and also for a.e. $Z_N \in \partial\mathcal{D}_N$. Let us finally define

$$W_N(Z_N) = W_N^{(\tilde{i}_N(Z_N), \tilde{j}_N(Z_N))}(Z_N) \quad (13)$$

so that $W_N(Z_N)$ is always equal to the correct collision parameter $|\omega \cdot (v_j - v_i)|$ globally along $\partial\mathcal{D}_N$.

The “proof” of the spacetime estimate is based on the following observation: the collision sum \sum_k on the left-hand side of (11) may be re-cast as an integral in time:

$$\sum_k W_N(\psi_N^{t_k} Z_N) \equiv \int_{\mathbb{R}} \delta_{\psi_N^t Z_N \in \partial\mathcal{D}_N} W_N(\psi_N^t Z_N) dt$$

Average both sides with respect to $f_N(0, Z_N) dZ_N$.

$$\begin{aligned} \int_{\mathcal{D}_N} \left\{ \sum_k W_N(\psi_N^{t_k} Z_N) \right\} f_N(0, Z_N) dZ_N &= \\ &= \int_{\mathcal{D}_N} \int_{\mathbb{R}} \delta_{\psi_N^t Z_N \in \partial\mathcal{D}_N} W_N(\psi_N^t Z_N) f_N(0, Z_N) dt dZ_N \end{aligned}$$

The double integral on the right-hand side reduces (by Fubini) to an integral of “something” over $\partial\mathcal{D}_N$, due to the delta-function and the identity $f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N)$. Unfortunately, making the change of variables precise requires a technical application of the divergence theorem and careful manipulation of delta functions. (The proof of [3] avoids any mention of delta functions.) Here we record the result of correct manipulations:

$$\begin{aligned} \int_{\mathcal{D}_N} \left\{ \sum_k W_N(\psi_N^{t_k} Z_N) \right\} f_N(0, Z_N) dZ_N &= \\ &= \int_{\mathbb{R}} \int_{\partial\mathcal{D}_N} [W_N(Z_N)]^2 f_N(t, Z_N) d\sigma_N dV_N dt \end{aligned} \quad (14)$$

where $d\sigma_N dV_N$ represents the surface measure on $\partial\mathcal{D}_N$.

To conclude, we bound the left hand side of (14) using Corollary 3.4, then reduce both sides using the symmetry of $f_N(t)$ and the definition of the marginals of $f_N(t)$. We have also simplified the right-hand side by optimal choice of the parameter $\lambda > 0$.

Proposition 4.1. *For each $N \in \mathbb{N}$, let $f_N(0)$ be an initial probability density on \mathcal{D}_N , which we assume to be symmetric under particle interchange, and let $f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N)$. Let $f_N^{(s)}(t)$, $1 \leq s \leq N$, denote the s -marginal of $f_N(t)$. Further assume that $f_N(0)$ is smooth and compactly supported in the interior of \mathcal{D}_N . Then for all $2 \leq s \leq N$ there holds*

$$\begin{aligned} &\sum_{1 \leq i < j \leq s} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{ds} \times \mathbb{R}^{d(s-1)} \times \mathbb{S}^{d-1}} |\omega \cdot (v_j - v_i)|^2 \times \\ &\quad \times f_N^{(s)}(t, \dots, x_i, v_i, \dots, x_i + \omega, v_j, \dots) d\omega dX_s^{(j)} dV_s dt \leq \\ &\leq C_d \frac{s(s-1)}{N} \left(\int |x|^2 f_N^{(1)}(0, x, v) dx dv \right)^{\frac{1}{2}} \left(\int |v|^2 f_N^{(1)}(0, x, v) dx dv \right)^{\frac{1}{2}} \end{aligned} \quad (15)$$

where $dX_s^{(j)} = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_s$ and C_d is a constant depending only on the dimension d .

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REFERENCES

- [1] R. K. Alexander, *The infinite hard-sphere system*, Ph.D. Thesis, 1975.
- [2] C. Cercignani, V. I. Gerasimenko, and D. Ya. Petrina, *Many-particle dynamics and kinetic equations*, Kluwer Academic Publishers, 1997.
- [3] R. Denlinger, *The propagation of chaos for a rarefied gas of hard spheres in vacuum*, Ph.D. Thesis, 2016.

- [4] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann: Hard spheres and short-range potentials*, Zurich Lec. Adv. Math. (2014).
- [5] R. Illner, *On the number of collisions in a hard sphere particle system in all space*, Transport Theory and Stat. Phys. **18** (1989), no. 1, 71–86.
- [6] R. Illner and M. Pulvirenti, *Global validity of the Boltzmann equation for two- and three-dimensional rare gas in vacuum: Erratum and improved result*, Comm. Math. Phys. **121** (1989), no. 1, 143–146.
- [7] M. Pulvirenti and S. Simonella, *On the evolution of the empirical measures for the hard-sphere dynamics*, arXiv:1504.03215 (2015).
- [8] S. Simonella, *Evolution of correlation functions in the hard sphere dynamics*, Journal of Statistical Physics **155** (2014), no. 6, 1191–1221.